L'Hospital's Rule

Guillaume-François-Antoine de L'Hospital, Marquis de Sainte-Mesme, Comte d'Entremont was born in Paris in 1661 and died there in 1704. At the age of fifteen, L'Hospital surprised his elders with his mathematical talent when he solved one of Pascal's problems on the cycloid. As a wealthy French nobleman, it was natural that L'Hospital would enter the cavalry. He rose to the rank of captain, but then took advantage of his wealth and social position and resigned to devote the rest of his life to the pursuit of mathematics. Lest you think this terribly altruistic, I should point out that his exceptional nearsightedness was a severe handicap for a cavalry officer. Little is known about l'Hospital as an individual, but "According to the testimony of his contemporaries, L'Hospital possessed a very attractive personality [being], among other things, modest and generous, two qualities which were not widespread among mathematicians of his time."[1]

L'Hospital retired to Paris where, by 1689, he became a member of the scientific circle of Nicolas Malebranche (1638–1715), a diligent and enthusiastic amateur mathematician. Others mathematicians involved were Carré, Reyneau, and Varignon. There is no doubt that L'Hospital was the most capable mathematician in this group and was probably the best French mathematician of his day.

When Johann Bernoulli (1667–1748) visited Paris in the fall of 1691 he quickly impressed the group with his knowledge of the new calculus of Leibniz. Especially impressive was the formula he possessed for finding the curvature of curves. He neglected to point out that the formula was due to his older brother Jakob. The two Bernoulli boys had studied the two calculus papers of Leibniz that had appeared in *Acta eruditorum*, his 1684 paper on the differential calculus and the 1686 paper on the integral calculus. These papers were extremely densely written, full of misprints, and deliberately obscure at critical points (to forestall criticism of his work, Leibniz made his differentials finite quantities rather than infinitesimals). Jakob Bernoulli had written to Leibniz for clarification, but Leibniz was away when the letter arrived. By the time a response was received the two brothers had mastered the new calculus on their own. This accomplishment is a true indication of their genius. The mathematicians in France, led by L'Hospital, had made some inroads into the differential calculus paper, but were completely baffled by that on the integral calculus. Thus it is not surprising that the 24 year old Johann Bernoulli

Guillaume François Marquis de l'Hôpital
1661–1704

[1] Abraham Robinson (1918–1974), "L'Hospital," *Dictionary of Scientific Biography*, vol. 8 (1973), pp. 304–305, esp. p. 305. Also see Robinson's paper "Concerning the history of the calculus," in his *Non-Standard Analysis*, North-Holland, 1966, pp. 260–282, which discusses the foundations of L'Hospital's work, especially his use of infinitesimals.

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made a big splash when he arrived in Paris in 1691.

It was not long before the Bernoulli boys progressed from fellow students of the calculus to rival researchers. No full study of the interactions and clashes of the Bernoulli brothers is available, but their rivalry soon spilled over into print. Let me give just one quotation to show the personality of the younger brother, Johann. The first independent work of Johann Bernoulli was to solve the catenary problem, i.e., to determine the shape of an elastic cable that is suspended between two points under its own weight.[2] He was extremely proud that he had solved this problem and that his brother Jakob, who had posed it, had not. Writing to Pierre Remond de Montmort (1678–1719) years later, on 29 September 1718, he boasted:

> The efforts of my brother were without success; for my part, I was more fortunate, for I found the skill (I say it without boasting, why should I conceal the truth?) to solve it in full and to reduce it to the rectification of the parabola. It is true that it cost me study that robbed me of rest for an entire night. It was much for those days and for the slight age and practice I then had, but the next morning, filled with joy, I ran to my brother, who was still struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more to try to prove the identity of the catenary with the parabola, since it is entirely false. The parabola indeed serves in the construction of the catenary, but the two curves are so different that one is algebraic, the other is transcendental.[3]

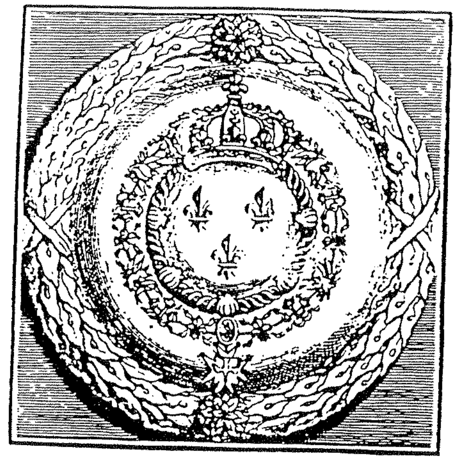
In 1691, when Johann Bernoulli (1667–1748) met L'Hospital, who was at the time particularly interested in learning about the new calculus of Leibniz, he again took advantage of his wealth and hired Bernoulli to teach him. This was a significant event for both of them. Bernoulli, who

ANALYSE

DES

INFINIMENT PETITS,

Pour l'intelligence des lignes courbes.



A PARIS,

DE L'IMPRIMERIE ROYALE.

M. DC. XCVI.

was newly married and unemployed, spent four months tutoring L'Hospital in the calculus, first in Paris and later at L'Hosptial's country estate of Ouques. In return, L'Hospital, with assistance from his friend Christiaan Huygens, obtained a university position for Bernoulli at Groningen in the Netherlands in 1695.

Today L'Hospital's name is only associated with the rule for evaluating limits of indeterminate forms, but in his day and for several generations thereafter, L'Hospital's fame rested on his book *Analyse des infiniment petits, pour l'intelligence des lignes courbes*, which was published anonymously in Paris in 1696. The title of this book can be very loosely translated as "Analysis using Infinitesimals for the Study of Curved Lines." Note that it refers to the study of curves, not functions; the function concept had not yet been introduced into mathematics (Euler did it in the next century). This was the first textbook on the differential calculus, as so deserves our special attention.

Toward the end of the seventeenth-century

[2] *Acta eruditorum*, 1691.

[3] *Der Briefwechsel von Johann Bernoulli, (1667–1748)*, Band I, Basel: Birkhuser, 531 pp., esp. 235–236, edited by Otto Spiess. This work the originals of the L'Hospital-Bernoulli correspondence and a detailed account of the relations between them. It also has a complete list of L'Hospital's publications. The quotation above comes from pp. 97–98; the translation from Morris Kline (1972–??), *Mathematical Thought from Ancient to Modern Times* (1972), Oxford, p. 473.

SECTION IX.

Solution de quelques Problémes qui dépendent des Méthodes précédentes.

PROPOSITION I.

Problême.

163. SOIT *une ligne courbe* AMD (AP = x, PM = y, AB = a) *telle que la valeur de l'appliquée* y *soit exprimée par une fraction, dont le numérateur & le dénominateur deviennent chacun zero lorsque* x = a, *c'est à dire lorsque le point* P *tombe sur le point donné* B. *On demande quelle doit être alors la valeur de l'appliquée* BD. FIG. 130.

Soient entendues deux lignes courbes *ANB*, *COB*, qui ayent pour axe commun la ligne *AB*, & qui soient telles que l'appliquée *PN* exprime le numérateur, & l'appliquée *PO* le dénominateur de la fraction générale qui convient à toutes les *PM*: de sorte que $PM = \frac{AB \times PN}{PO}$. Il est clair que ces deux courbes se rencontreront au point *B*; puisque par la supposition *PN* & *PO* deviennent chacune zero lorsque le point *P* tombe en *B*. Cela posé, si l'on imagine une appliquée *bd* infiniment proche de *BD*, & qui rencontre les lignes courbes *ANB*, *COB* aux points *f*, *g*; l'on aura $bd = \frac{AB \times bf}{bg}$, laquelle * ne différe pas de *BD*. * *Art.* 2.
Il n'est donc question que de trouver le rapport de *bg* à *bf*. Or il est visible que la coupée *AP* devenant *AB*, les appliquées *PN*, *PO* deviennent nulles, & que *AP* devenant *Ab*, elles deviennent *bf*, *bg*. D'où il suit que ces appliquées, elles mêmes *bf*, *bg*, sont la différence des appliquées en *B* & *b* par rapport aux courbes *ANB*, *COB*; & partant que si l'on prend la différence du numérateur, & qu'on la divise par la différence du dénominateur, après

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the many papers of Leibniz and the Bernoulli's in *Acta eruditorum* and the *Journal des sçavans* generated a great deal of enthusiasm for the new calculus, but if you were not a genius of their stature, these papers were impenetrable. Consequently l'Hospital's book was a great success. Seldom has a book been so well received for it truly filled a need, something few textbook writers can say of their books today. It is not surprising then that the book went through numerous reprintings and editions in the eighteenth-century. L'Hospital's name appeared on the title page of the second edition of 1715, which appeared after his death. It was even translated into English in 1730 by E. Stone (indicating that the rift between the English and continental mathematicians was not that strong), but Newton's fluxional notation was used in this edition.[4]

L'Hospital did make other interesting contributions to mathematics, especially in his posthumous book on the conics, *Traté analytique des sectiones coniques* (Paris, 1707), but none were so great as the expository work in his *Analyse*.[5]

In the preface to his book, just after mentioning Leibniz and especially "the young professor at Groningen," i.e., Bernoulli, L'Hospital admitted "I have made free use of their discoveries, so that I frankly return to them whatever they please to claim as their own." He does not spell out precisely what he learned from them, but of course it was common knowledge that Leibniz had published the first papers on the calculus and that Bernoulli had tutored L'Hospital. He does explicitly state that the foundations of the calculus given in the book are his own discovery, and he was also instrumental in getting Leibniz to make statements about his own foundational views.

After Johannn Bernoulli received a copy in Groningen, he wrote to L'Hospital praising the book as admirably done, praising the arrangement of the propositions, and praising the intelligibility of the exposition, even thanking him for mentioning his name, and promising to return the favor in his next publication. These ingratiating comments seem out of character for Bernoulli, but there is truth in what he says: L'Hospital's exposition was deserving of praise. It is a wonderful book.

Immediately after L'Hospital's death in February 1704, Bernoulli published a generalization of L'Hospital's rule which allowed for its repeated application. In this paper, in the August issue of *Acta eruditorum*, he complained that L'Hospital had not given him ample credit and then laid public claim to the most novel and interesting result in the book, the theorem in §163 that we now call L'Hospital's rule. [Who instituted this name? When?] As Bernoulli was not noted for either modesty or generosity, and as he had already been involved in more than his share of priority disputes, this claim was generally dismissed by his contemporaries. While Bernoulli is a difficult character to defend, it should be said that his claim was only made after L'Hospital's friend Saurin implied that the rule was due to Leibniz.[6]

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4 For information on the various editions see Bernoulli, op. cit., 1955.

5 Julian Lowell Coolidge (1873–1954), "Guillaume L'Hospital, Marquis de Sainte-Mesme," pp. 147–170 in his *The Mathematics of Great Amateurs* (1949), Oxford: Clarendon Press. Dover reprinted this in 1963. This chapter is the best source of information about the entire corpus of L'Hospital's work. However, sometimes Coolidge is so brief as to be incomprehensible.

6 "Perfectio Regula sua edita in Libro Gall. *Anal-*

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The question of priority was only answered in this century when Johann Bernoulli's *Lectiones de calculo differentialium* [Lectures on the Differential Calculus] were published in 1922 by Paul Schafheitlin (1861–1924).[7] They were written in 1691–1692 when Bernoulli was in France tutoring L'Hospital and they contain considerable overlap with the material that appeared in L'Hospital's *Analyse des infiniment petits* of 1696. This publication raised the plagiarism issue again after two centuries, but did not clear things up entirely, for the historian Carl Boyer, writing a quarter of a millenium after L'Hospital's book appeared, still agreed with Eneström's 1894a opinion that "the broad claims of Bernoulli with respect to the authorship of the material are not substantiated."[8]

The situation was not cleared up completely until 1955 when Bernoulli's correspondence, including that with L'Hospital, was published. It contains a most unusual letter that L'Hospital, then in Paris, wrote to Bernoulli in his home town of Basel on 17 March 1694:

> I shall give you with pleasure a pension of three hundred livres, which will begin on the first of January of the present year, and I shall send two hundred livres for the first half of the year because of the journals that you have sent, and it will be one hundred and fifty livres for the other half of the year, and so in the future. I promise to increase this pension soon, since I know it to be very moderate, and I shall do this as soon as my affairs are a little less confused. . . . I am not so unreasonable as to ask for this all your time, but I shall ask you to give me occasionally some hours of your time to work on what I shall ask you and also to communicate to me your discoveries, with the request not to mention them to others. I also ask you to send neither to M[onsieur]. Varignon nor to others copies of the notes that you let me have, for it would not please me if they were made public. Send me your answer to all this and believe me, Monsieur tout à vous
> le M. de Lhospital

Bernoulli's response to this letter has been lost, but we know from his letter of 22 July 1694 that he accepted L'Hospital's proposal of a regular salary in exchange for help on mathematical problems and, implicitly, for providing L'Hospital with results to publish under his own name. We do not know how long this arrangement lasted, but Bernoulli's finances improved, and, as mentioned earlier, he soon had a position in Groningen. Considering Bernoulli's effusive praise of L'Hospital's book, it is likely that the agreement was still in effect in 1696. At any rate, the agreement prevented Bernoulli from laying claim to L'Hospital's rule while he was still alive. As we have seen though, he staked his claim as soon as L'Hospital died.

We can learn several interesting things from this letter. Both men kept a copy of the lectures which Bernoulli wrote while he was in France in 1691–1692. These lectures were the first exposition of the new calculus of Leibniz, and they are an excellent presentation. There were also lectures on the integral calculus which L'Hospital planned publish, but did not when he learned that Leibniz has similar intentions. Unfortunately, these plans of Leibniz, like many of his plans, came to nought. Bernoulli's lectures on the integral calculus were published within his own lifetime in his *Opera omnia* of 1742.[9] They contained a footnote that he had also written on the differential calculus

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yse des infiniment petits, Art. 163. pro determinando valore fractionis, cujus Numerator & Denominator certo casu evanescunt," *Acta eruditorum*, August 1704, pp. 375ff; reprinted as No. LXXI in his *Opera omnia*, vol. 1, pp. 401–405. The example that Bernoulli gives here provides a useful classroom example:

$$y = (a\sqrt{(4a^3 + 4x^3)} - ax - aa) : (\sqrt{(2aa + 2xx)} - x - a).$$

Bernoulli asks for the value when $x = a$; we seek the limit on y as x tends to a.

[7] *Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691–1692* (1924), Ostwald's Klassiker #211, 56 pp (Engelmann, Leipzig). Translation of 1922a edited by P. Schafheitlin. Reviewed by H. Wieleitner, *Isis*, vol. 5 (1923), pp. 186–7.

[8] Carl Benjamin Boyer (1906–1976), "The first calculus textbooks," *Mathematics Teacher*, 39 (1946), 159–167, especially p. 163. This is very useful, but remember it was written before Bernoulli 1955 appeared. It discusses the contents of L'Hospital's book.

[9] Johann Bernoulli, "Lectiones mathematicae, de methodo integralium, aliisque. Conscriptae in usum ill. Marchionis Hospitalii, cum auctor parisiis ageret, annis 1691 & 1692," pp. 385–558 of vol. 3 of his *Opera omnia*, four volumes, Lausanne and Geneva 1742; reprinted Hildesheim: Georg Olms, 1968. These lectures were written in Paris in 1691–

in 1691–1692, but that similar work was published by L'Hospital in his 1696 book. Unfortunately, it was too late for these lectures to have a significant effect on the development of the integral calculus.

The most important thing that we learn from the correspondence is that Bernoulli's claim to being the discoverer of L'Hospital's rule is correct, for in his letter of 22 July 1694 the rule appears along with two examples. One of these examples appears in L'Hospital's *Analyse des infiniment petits*, the other appears there in slightly modified form.

Now that we know that L'Hospital's rule is due to Johann Bernoulli, the question arises as to whether we should rename it. Of course not! For one thing, we could never change such a well established tradition in mathematics. Perhaps the only benefit of doing so would be that we would not have students talking about "The Hospital Rule"; instead they would refer to "The Bernoulli Rulie." Seriously though, this is an excellent example of the principle that one cannot lay claim to a scientific discovery until one has given it to the scientific community. Since it was L'Hospital who gave the rule to the world, he should get the credit. Besides, he bought it fair and square.

The Original Statement of L'Hospital's Rule

Without doubt the most famous section of L'Hospital's *Analyse* is §163, which contains the first printed statement and proof of the famous rule:

163 Proposition I. *Let AMD [Fig. 130] be a curve ($AP = x$, $PM = y$, $AB = a$) of such a nature, that the value of the ordinate y is expressed by a fraction, the numerator and denominator of which, do each of them become 0 when $x = a$, viz. when the point P coincides with the given point B. It is required to find what will be the value of the ordinate BD.*[10]

L'Hospital next gives a proof of the result and then follows it with two examples. We shall analyze the proof shortly, but first let us see his examples and make a few comments about them. The first asks for the evaluation of

$$y = \frac{\sqrt{2z^3 - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{x^3}}$$

when $x = a$. The solution provided is quite similar to what our students do today and so will not be reproduced [do it as an exercise]. He computes the quotient of the differentials and then sets $x = a$. Note that derivatives are not used; that concept is still in the future—see Grabiner 1983a. Naturally no limits are used here for that concept had to wait for Cauchy in the nineteenth century.

A Conjecture: It should be clear that this is a rather weird first example. Certainly I would not ordinarily begin with such a complicated example in class, although since this is "the first example," I do mention it as part of the curious history of L'Hospital's rule. Years ago I thought that this example must have arisen from some physical problem; this was when I still was inclined to believe the Kline-May thesis that all good (important, interesting, ... —add enough modifiers to make this statement true) mathematics arises out of real world problems. But I was unable to find any such origin for this example. Thus I concluded that Bernoulli simply concocted this example for use in the "classroom" with L'Hospital. Recently I found this same example in a letter from Bernoulli to L'Hospital [1955, p. 232]. In this letter Bernoulli mentions this expression in the same sentence in which he mentions the Florentine Enigma of Vincenzo Viviani (1622–1703): cut four congruent windows from a hemisphere in such a way that the area remaining is quadrable, i.e., the integral for this area can be evaluated in elementary (hopefully algebraic) terms. Unfortunately, the passage is quite obscure and I am unable to tell whether he is just listing two separate problems that he is working on or means to imply that they are closely connected. I rather think that he means the latter, for the two examples are

92 for the use of L'Hospital. See the footnote on p. 387 where he states that his lectures on the differential calculus were published by L'Hospital.

[10] Dirk J. Struik (born 1894), "L'Hôpital. The analysis of the infinitely small," pp. 312–316 of his *A Source Book in Mathemtics, 1200-1800*, Harvard University Press, 1969. Struik's "The origin of L'Hôpital's rule," *Mathematics Teacher*, 56 (1963), 257–260 is very informative. It quotes the letter of 17 March 1694 from L'Hospital to Johann Bernoulli from Bernouli 1955. Stuik's "The origin of L'Hospital's rule," pp. 435–439 in NCTM's Thirty-First Yearbook, *Historical Topics for the Mathematics Classroom*, 1969 contains information here that is not in either of the above. For information about Struik, see David E. Rowe, "Interview with Dirk Jan Struik," *The Mathematical Intelligencer*, 11 (1989), no. 1, pp. 14–26

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separated by a semicolon. The second example of L'Hospital is considerably simpler:

Example 2. Evaluate the following expression when $x = a$.

$$y = \frac{aa - ax}{a - \sqrt{ax}}$$

Curiously, L'Hospital does not write out this example using his new rule, but comments that "We could solve this example without need of the calculus of differentials in this way." He solves for the radical, squares to eliminate it, and then divides by $x - a$. Next he sets $x = a$. This is the old math. It would fit nicely into his last chapter where he compares the new methods with the old. Bernoulli's second example would do even better. This old technique, as well as its revision by Leibniz and Newton was later soundly — and rightly — criticized by Berkeley.

L'Hospital simplified this example a bit from that given by Bernoulli in his letter of 22 July 1694 (1955, p. 236):

$$y = \frac{a\sqrt{ax} - xx}{a - \sqrt{ax}}$$

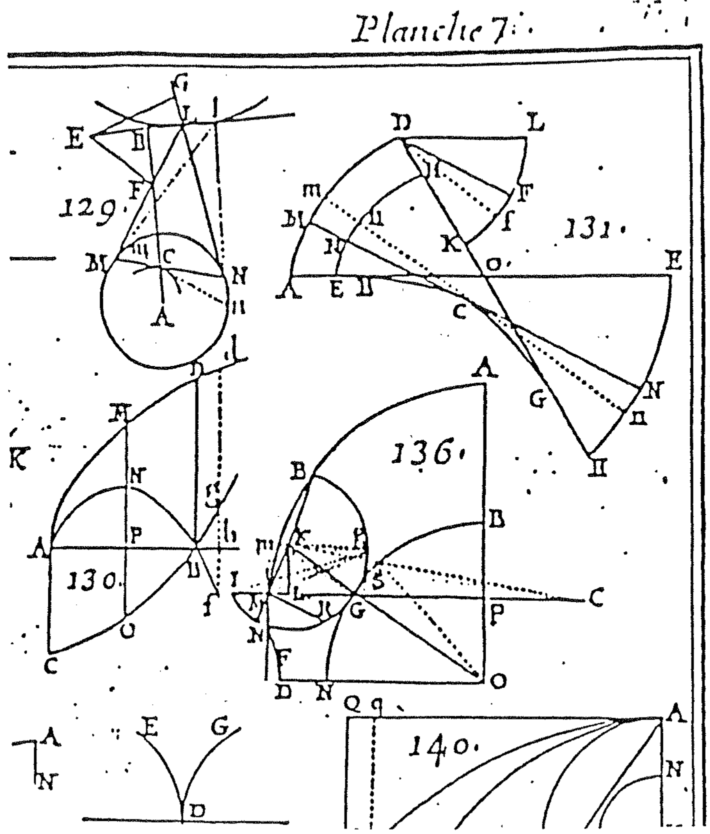
Now Bernoulli gives this example "for verifying the method" that he has just sent to L'Hospital. He points out that it can be worked out using the "common geometry" of Descartes that was well known at the time. Pedagogically this is a beter way of using the example; we should show our students what can (and cannot) be done using older methods and thus use these examples to justify newly introduced techniques. This example is harder to verify using the old method, for you obtain a quadratic equation when eliminating the roots [It is not clear to me why only one root is considered]. Thus we see that L'Hospital simplified the example, but not used it so cleverly.

These two examples are the only ones given, and L'Hospital's rule is applied to just one of them. It is as if L'Hospital were saying, "OK, that's it; if you don't get it with these two examples, too bad; it's time to move on to the next section." While I am not advocating greater speed in our courses today, perhaps there is a pedagogical point here that we should heed. There is no need to give numerous examples in order to get the idea across. We concentrate too much on computational technique and not enough on the underlying ideas.

Perhaps you are surprised that there are no examples dealing with exponentials, logarithms, and the trigonometric functions. The explanation is historical: Johann Bernoulli was doing the first work on calculus of exponentials at that very time. The calculus of the trigonometric functions, and indeed the trigonometric functions themselves (*qua* functions), were not developed until the 1730s when Euler did so.

The Proof of L'Hospital's Rule

Figure 130 is on a plate at the back of the volume, but we reproduce part of it here for the student's convenience and amusement:



Next we shall give the proof in L'Hospital's book:

> Let ANB, COB, be two curves (having the line AB as a common axis) of such a nature, that the ordinate PN expresses the numerator, and the ordinate PO the denominator of the general fraction representing any ordinate PM: so that $PM = (ABxPN)/PO$. Then it is manifest, that these two curves will meet one another in the point B; since by the supposition PN, PO do each become 0 when the point P falls in B. This being supposed, if an ordinate bd be imagined infinitely near to BD, cutting the curves ANB, COB in

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the points f, g; then will $bd = (ABxbf)/bg$, which will be equal to BD. Now our business is only to find the relation of bg to bf. In order thereto it is manifest, when the abscissa AP becomes AB, the ordinates PN, PO will be 0, and when AP becomes Ab, they do become bf, bg. Whence it follows, that he said ordinates bf, bg, themselves, are the differentials of the ordinates in B and b, with regard to the curves ANB, COB; and consequently, if the differential of the numerator be found, and that be divided by the differential of the denominator, and having made $x = a = Ab$ or AB, we shall have the value of the ordinates bd or BD sought. Which was to be found.[11]

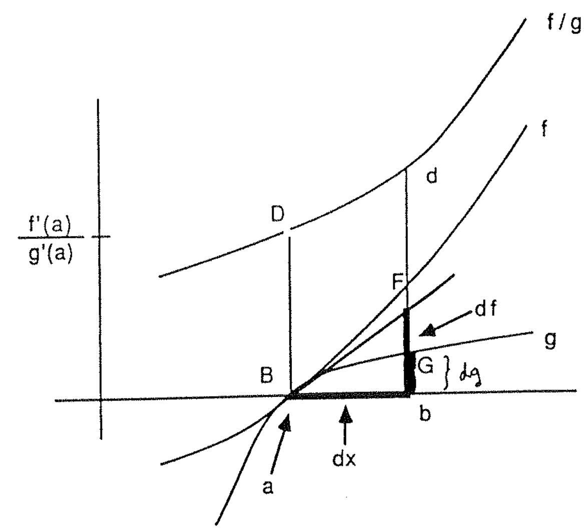
The first thing to realize is that L'Hospital is dealing with curves, not functions. The calculus of Newton and Leibniz was a calculus of curves; Euler introduced a calculus of functions in his *Introductio in analysin infinitorum* (1748). For presentation today, we certainly want to state this in terms of functions.

L'Hospital's diagram is somewhat hard for the modern reader to decipher in that L'Hospital (and his contemporaries) does not know which way is up. In his diagram both PM and PO represent positive quantities. The curve ANB is above the axis AB and the curve COB is below, so the quotient should be below in our way of doing analytic geometry. But this is not so for L'Hospital. His quotient curve AMD is above the axis. To avoid this difficulty we shall redraw the picture:

In L'Hospital's diagram the curve ANB was above the axis, but we moved it in our diagram so that it looks more familiar to us. It is now the curve f, or rather the graph of the function f. The curve COB has been relabelled g. Now suppose the two curves f and g which meet at the point a, where both are 0. The quotient f/g is not defined for $x = a$, but it is this value that L'Hospital wishes to determine. What L'Hospital is saying is that

$$BD \approx bd = \frac{bf}{bG} \approx \frac{df}{dg} = \frac{df/dx}{dg/dx} = \frac{f'(a)}{g'(a)}.$$

Thus he provides a very nice proof of the rule. It is much more intuitive than the proofs that are customarily given today using Cauchy's extended



$$BD \approx bd = \frac{bF}{bG} \approx \frac{df}{dg} = \frac{df/dx}{dg/dx} = \frac{f'(a)}{g'(a)}$$

mean value theorem.[12] Admittedly, infinitesimals are used in the proof, but, after the work of Abraham Robinson in the 1960s we need no longer be concerned about the rigor of infinitesimal techniques. In my view, what is lost in rigor is more than made up for in intuitiveness and understanding.

Moral: The original proofs are often much more understandable than modern ones.

[11] Struik, *Source Book*, p. 316.

[12] Yes, I am perfectly aware that this proof admits of counterexamples. It does not work when $bG = 0$. More precisely, if the function in the denominator is 0 arbitrarily close to a, then the proof fails. But, in the seventeenth-century, there were no such functions. Today, if one of my students comes to me with a function such as $\sin(1/x)$, then we are ready to do some more mathematics.